Journal of Approximation Theory **96**, 171–181 (1999)
Article ID jath.1997.3240, available online at http://www.idealibrary.com on IDEAL®

Polynomial Projections in $C[-1, 1]$ and $L^1(-1, 1)$ with Growth n^γ, $0<\gamma\leqslant 1/2$

Ernst Görlich and Alfred P. Rohs*

Lehrstuhl A für Mathematik, RWTH Aachen, 52056 Aachen, Germany
E-mail: goerlich@rwth-aachen.de

Communicated by E. W. Cheney

Received November 12, 1996; accepted in revised form October 29, 1997

For $\gamma\in(0, 1/2]$ we construct n-dimensional polynomial subspaces Y_n of $C[-1, 1]$ and $L^1(-1, 1)$ such that the relative projection constants $\lambda(Y_n, C[-1, 1])$ and $\lambda(Y_n, L^1(-1, 1))$ grow as n^γ. These subspaces are spanned by Chebyshev polynomials of the first and second kind, respectively. The spaces $L^1_{w(\alpha, \beta)}$ where $w_{\alpha, \beta}$ is the weight function of the Jacobi polynomials and $(\alpha, \beta)\in\{(-1/2, -1/2), (-1/2, 0), (0, -1/2)\}$ are also studied.

1. INTRODUCTION AND MAIN RESULT

The operator norms of the Fourier projections from one of the spaces $C_{2\pi}$ and $L^1_{2\pi}$ of 2π-periodic functions onto an n-dimensional subspace of the form $Y_n=\operatorname{span}\{e^{ik_jx}; 1\leqslant j\leqslant n\}$ with integers $k_1<k_2<\cdots<k_n$ can grow like n^γ as $n\to\infty$ for any $\gamma\in(0, 1/2)$. This was shown by Shekhtman [10] by explicit construction of sequences $\{k_j\}$.

In view of the minimality of the Fourier projections, $\lambda(Y_n, C_{2\pi})$ and $\lambda(Y_n, L^1_{2\pi})$ also grow as n^γ with the usual definition

$$\lambda(Y, X):=\inf\{\|P\|_{[X]};\ P \text{ is a continuous linear projection from } X \text{ onto } Y\} \tag{1}$$

of the relative projection constant of a linear subspace Y in a Banach space X with norm $\|\cdot\|_X$ and operator norm $\|\cdot\|_{[X]}$.

The corresponding question for the spaces $C[-1, 1]$ and $L^1(-1, 1)$, which is the subject of the present paper, depends in an essential way on the choice of functions which span Y_n. It is not useful to choose $Y_n=\operatorname{span}\{x^{k_j}; 1\leqslant j\leqslant n\}$ since for a sufficiently sparse sequence $\{k_j\}$, $\lambda(Y_n, C[-1,])$ and

* The second named author was supported by the Graduiertenkolleg "Analyse und Konstruktion in der Mathematik," RWTH Aachen.

0021-9045/99 $30.00

$\lambda(Y_n, L^1(-1, 1))$ may be bounded as $n \to \infty$; see Newman and Shekhtman [9, Theorem 1] and the present authors [7, Theorem 2], respectively.

In fact, the Shekhtman construction in [10], which relies on the interplay between lacunary and dense segments in the sequence $\{k_j\}$, just increases the influence of the lacunary segments in order to achieve the larger rates.

Instead of a span of monomials, or a span of Legendre polynomials, our choice will be

$$Y_n := \operatorname{span}\{P_{k_j}^{\alpha, \beta}(x);\ 1 \leqslant j \leqslant n\}$$

with $(\alpha, \beta) = (1/2, 1/2)$ in case of $L^1(-1, 1)$ and $(\alpha, \beta) = (-1/2, -1/2)$ in case of $C[-1, 1]$. Here $P_m^{\alpha, \beta}(x)$ denotes the Jacobi polynomials normalized by $P_m^{\alpha, \beta}(1) = \binom{m+\alpha}{m}$. This makes it possible on the one hand to employ a special Berman–Marcinkiewicz type identity (9) which draws a connection between four L^1-spaces with particular Jacobi weights $w_{\alpha, \beta}(x) = (1-x)^\alpha (1+x)^\beta$ and the Jacobi partial sum operators $S_n^{2\alpha + 1/2, 2\beta + 1/2}$, and on the other, to show that the sparse sequences $\{k_j\}$ furnish the larger operator norms again (Lemmas 3 and 5). As a consequence, the essentials of Shekhtman's construction remain applicable in the proof of our main result.

THEOREM. *Let $X(-1, 1)$ be one of the spaces $C[-1, 1]$ or $L^1_{w(\alpha, \beta)}$ with $(\alpha, \beta) \in \{(0, 0),\ (-1/2, -1/2),\ (-1/2, 0),\ (0, -1/2)\}$. For each $\gamma \in (0, 1/2)$ there exists a set of integers $0 \leqslant k_1 < \cdots < k_n$ and a subspace $Y_n := \operatorname{span}\{P_{k_j}^{2\alpha + 1/2, 2\beta + 1/2}(x);\ 1 \leqslant j \leqslant n\}$ such that*

$$C_1 n^\gamma \leqslant \lambda(Y_n, X(-1, 1)) \leqslant C_2 n^\gamma \tag{2}$$

for certain constants $C_1, C_2 > 0$ that do not depend on n.

Here $L^1_{w(\alpha, \beta)}$ denotes the space of complex valued functions f on $(-1, 1)$ with $\|f\|_{L^1_{w(\alpha, \beta)}} := \int_{-1}^{1} |f(x)|\, w_{\alpha, \beta}(x)\, dx < \infty$.

The paper concludes with an example (Remark 10) showing that the mere comparison of the growth rate of two sequences $\{k_j\}$ does not allow one to predict which of the two projection constants increases faster.

2. PROOFS

A set of the form

$$I = \{k_j \in \mathbb{N}_0;\ k_1 < k_2 < \cdots,\ j \in \mathbb{N}\} \tag{3}$$

will be called an index set, and I is called lacunary if $k_1 \geqslant 1$ and $k_{j+1}/k_j \geqslant r > 1$, $j \in \mathbb{N}$.

DEFINITION 1. Let I be an index set. For any $m \in \mathbb{N}$ define

$$\operatorname{num}(m) := \#\{(i, j) \in \mathbb{N} \times \mathbb{N}; m = k_i + k_j \text{ or } m = |k_i - k_j|, k_i, k_j \in I\},$$

where $\# M$ denotes the cardinality of a set M. The set I is called thin if there exists a constant C such that $\operatorname{num}(m) \leqslant C$ for all $m \in \mathbb{N}$.

PROPOSITION 2. *A lacunary index set is thin. The converse is not valid.*

Proof. The number $\operatorname{num}(m)$ of representations of a given $m \in \mathbb{N}$ as a sum or a difference of elements of I does not exceed twice the number of representations $m = k_i + k_j$ with $i \geqslant j$ or $m = k_\nu - k_\mu$ with $\nu \geqslant \mu$. If $m = k_i + k_j$ and $i \geqslant j$, then $2k_i \geqslant m \geqslant k_i$. Thus there can be at most σ such representations, where σ is the number of elements $k_i \in I$ with $m/2 \leqslant k_i \leqslant m$. In view of the lacunarity of I we have $2 \geqslant r^{\sigma - 1}$, i.e., $\sigma \leqslant [1 + \log 2/\log r]$, where $[a]$ denotes the integer part of a. Similarly, $\delta \leqslant [2 - \log(r-1)/\log r]$ is obtained for the number δ of representations $m = k_\nu - k_\mu$ with $\nu \geqslant \mu$, and it follows that $\operatorname{mum}(m) \leqslant 2(\sigma + \delta) =: C$.

As to the converse, a thin index set with $k_j \leqslant j^4$, $j \in \mathbb{N}$, is clearly not lacunary. To construct such a set we start with a fixed $\nu \in \mathbb{N}$ and numbers $k_1 < k_2 < \cdots < k_\nu$ such that $k_j \leqslant j^4$, $1 \leqslant j \leqslant \nu$. The set

$$M_\nu := \{m \in \mathbb{N}; m = k_i + k_j \text{ or } m = |k_i - k_j|, 1 \leqslant i, j \leqslant \nu\}$$

of all sums and differences of pairs of k_j satisfies $\# M_\nu \leqslant \nu(\nu+1)$, and each element of M_ν has at most C different representations for some constant $C \in \mathbb{N}$. If the next index $k_{\nu+1}$ can be chosen such that

$$k_{\nu+1} \in \{\nu^4 + 1, ..., (\nu+1)^4\} =: A_{\nu+1} \quad \text{and} \quad k_{\nu+1} \pm k_j \notin M_\nu, 1 \leqslant j \leqslant \nu, \tag{4}$$

the numbers of representations for elements of $M_{\nu+1}$ remains bounded by C. Such a choice is possible since

$$k_{\nu+1} \notin \{m \pm k_j; m \in M_\nu, 1 \leqslant j \leqslant \nu\} =: B_{\nu+1},$$

i.e., there exists $\# A_{\nu+1} - \# B_{\nu+1} > 2\nu^3$ possibilities.

Continuing this process, one obtains a sequence $\{k_j\}_{j \in \mathbb{N}}$ such that each $m \in \bigcup_{j \geqslant \nu} M_j$ has no more than C different representations as sums or differences of previous elements. Any $m \in \mathbb{N}_0 \backslash \bigcup_{j \geqslant \nu} M_j$ has no representation at all, and the proof is complete. ∎

For an index set of type (3) let $I_n = \{k_j \in I;\ 1 \leqslant j \leqslant n\}$ and denote by $S_{I_n}^{\alpha,\beta}$ the Jacobi partial sum operator with respect to the set $\{P_{k_j}^{\alpha,\beta}(x);\ k_j \in I_n\}$.

LEMMA 3. *Let* $(\alpha, \beta) \in \{(0,0),\ (-1/2, -1/2),\ (-1/2, 0),\ (0, -1/2)\}$, *and let I be a thin index set. There exists a constant* $C > 0$, *independent of* n, *such that*

$$\|S_{I_n}^{2\alpha+1/2,\, 2\beta+1/2}\|_{[L^1_{w(\alpha,\beta)}]} \geqslant C\sqrt{n}. \tag{5}$$

Proof. Denoting the orthonormal Jacobi polynomials by $\tilde{P}_m^{2\alpha+1/2,\, 2\beta+1/2}(x)$, one has

$$\|S_{I_n}^{2\alpha+1/2,\, 2\beta+1/2}\|_{[L^1_{w(\alpha,\beta)}]} = \operatorname*{ess\,sup}_{t \in (-1,1)} \int_{-1}^{1} |K_{I_n}^{\alpha,\beta}(x,t)|\, w_{-1/2,\,-1/2}(x)\, dx,$$

$$\begin{aligned} K_{I_n}^{\alpha,\beta}(x,t) = \sum_{j=1}^{n} &\tilde{P}_{k_j}^{2\alpha+1/2,\, 2\beta+1/2}(x)\, w_{\alpha+1/2,\, \beta+1/2}(x) \\ &\times \tilde{P}_{k_j}^{2\alpha+1/2,\, 2\beta+1/2}(t)\, w_{\alpha+1/2,\, \beta+1/2}(t). \end{aligned}$$

Following a pattern in DeVore and Lorentz [2, p. 284], the function num(m) of Definition 1 is brought into play via the forth power of the kernel. Hölder's inequality with $p = 3$ yields

$$\begin{aligned} &\operatorname*{ess\,sup}_{t\in(-1,1)} \|(K_{I_n}^{\alpha,\beta}(\cdot,t))^2\|_{L^1_{w(-1/2,\,-1/2)}} \\ &\quad\leqslant \Big\{\operatorname*{ess\,sup}_{t\in(-1,1)} \|(K_{I_n}^{\alpha,\beta}(\cdot,t))^4\|_{L^1_{w(-1/2,\,-1/2)}}\Big\}^{1/3} \|S_{I_n}^{2\alpha+1/2,\, 2\beta+1/2}\|^{2/3}_{[L^1_{w(\alpha,\beta)}]}. \end{aligned} \tag{6}$$

The left hand side is

$$\begin{aligned} &\operatorname*{ess\,sup}_{t\in(-1,1)} \left| \sum_{i=1}^{n} (\tilde{P}_{k_i}^{2\alpha+1/2,\, 2\beta+1/2}(t))^2\, w_{2\alpha+1,\, 2\beta+1}(t) \right| \\ &\qquad \geqslant \frac{1}{\pi} \int_{-1}^{1} \left| \sum \cdots \right| w_{-1/2,\,-1/2}(t)\, dt = \frac{n}{\pi}. \end{aligned}$$

To estimate the right hand side, let $(\alpha, \beta) = (0, 0)$ first. Using the representation

$$\tilde{P}_m^{1/2,\, 1/2}(x) = \sqrt{\frac{2}{\pi}} \sin((m+1) \arccos x)(1-x^2)^{-1/2}, \tag{7}$$

one obtains

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t\in(-1,1)} \|(K_{I_n}^{0,0}(\cdot,t))^4\|_{L^1_{w(-1/2,\,-1/2)}} \\
&\quad = \frac{1}{\pi^4} \max_{\psi\in[0,\pi]} \int_0^\pi \Bigg[4 \sum_{i,\,j=1}^{n} \sin((k_i+1)\,\psi)\,\sin((k_j+1)\,\psi) \\
&\qquad \times \sin((k_i+1)\,\varphi)\,\sin((k_j+1)\,\varphi)\Bigg]^2\,d\varphi \\
&\quad \leqslant \frac{1}{\pi^4} \sum_{i,\,j,\,l,\,m=1}^{n} \Big[\max_{\psi\in[0,\pi]} |\cos((k_i-k_j)\,\psi)-\cos((k_i+k_j+2)\,\psi)| \\
&\qquad \times |\cos((k_l-k_m)\,\psi)-\cos((k_l+k_m+2)\,\psi)|\Big] \\
&\qquad \times \Bigg\{\bigg|\int_0^\pi \cos((k_i-k_j)\,\varphi)\cos((k_l-k_m)\,\varphi)\,d\varphi\bigg| \\
&\qquad + \bigg|\int_0^\pi \cos((k_i-k_j)\,\varphi)\cos((k_l+k_m+2)\,\varphi)\,d\varphi\bigg| \\
&\qquad + \bigg|\int_0^\pi \cos((k_i+k_j+2)\,\varphi)\cos((k_l-k_m)\,\varphi)\,d\varphi\bigg| \\
&\qquad + \bigg|\int_0^\pi \cos((k_i+k_j+2)\,\varphi)\cos((k_l+k_m+2)\,\varphi)\,d\varphi\bigg|\Bigg\} \\
&\quad \leqslant \frac{4}{\pi^4}\left(\sum_1+\sum_2+\sum_3+\sum_4\right),
\end{aligned}
$$

say, where the term in square brackets has been estimated by 4. Since I is thin, it follows that

$$
\sum_1 = \sum_{\substack{i,\,j,\,l,\,m=1\\ k_i-k_j=k_l-k_m=0}}^{n} \pi + \sum_{\substack{i,\,j,\,l,\,m=1\\ k_i-k_j=k_l-k_m\neq 0}}^{n} \frac{\pi}{2} \leqslant \pi n^2 + \frac{\pi}{2}\sum_{i,\,j=1}^{n} \operatorname{num}(k_i-k_j) \leqslant C_1 n^2,
$$

and similarly $\sum_i \leqslant C_i n^2$, $2\leqslant i\leqslant 4$. Hence

$$
\operatorname*{ess\,sup}_{t\in(-1,1)} \|(K_{I_n}^{0,0}(\cdot,t))^4\|_{L^1_{w(-1/2,\,-1/2)}} \leqslant C_5 n^2.
$$

The cases $(\alpha, \beta) = (-1/2, -1/2)$ and $(-1/2, 0)$ are treated analogously, using the representations

$$\tilde{P}_m^{-1/2,\,-1/2}(x) = \begin{cases} \sqrt{2/\pi}\cos(m \arccos x), & m \in \mathbb{N} \\ 1/\sqrt{\pi}, & m = 0, \end{cases}$$

$$\tilde{P}_m^{-1/2,\,1/2}(x) = \sqrt{2/\pi}\cos((m+1/2)\arccos x)(1+x)^{-1/2}.$$

The case $(\alpha, \beta) = (0, -1/2)$ follows by symmetry.

Finally, (6) yields

$$\| S_{I_n}^{2\alpha+1/2,\,2\beta+1/2} \|_{[L^1_{w(\alpha,\beta)}]} \geqslant \left[\frac{n/\pi}{(C_5 n^2)^{1/3}} \right]^{3/2}. \quad \blacksquare$$

Definition 4. For $(\alpha, \beta) \in \{(0, 0), (-1/2, -1/2), (-1/2, 0), (0, -1/2)\}$ the generalized translation operator $\tau_t^{\alpha,\beta}: L^1_{w(\alpha,\beta)} \to L^1_{w(\alpha,\beta)}$ is defined by

$$(\tau_t^{\alpha,\beta} f)(x) := (\tau_t(f(\cdot)\, w_{\alpha+1/2,\,\beta+1/2}(\cdot)))(x)\, w_{-\alpha-1/2,\,-\beta-1/2}(x), \tag{8}$$

$$\begin{aligned} (\tau_t g)(x) &:= \tfrac{1}{2}[\, g(\cos(\varphi(x,t)))\, a(\varphi(x,t)) + g(\cos(\psi(x,t)))\, a(\psi(x,t))] \\ &= \tfrac{1}{2}[\, g(xt + \sqrt{1-x^2}\sqrt{1-t^2})\, a(\varphi(x,t)) \\ &\quad + g(xt - \sqrt{1-x^2}\sqrt{1-t^2})\, a(\psi(x,t))] \qquad (x, t \in [-1, 1]) \end{aligned}$$

with

$$a(\theta) := \begin{cases} \operatorname{sign}(\sin\theta), & (\alpha,\beta) = (0,0) \\ 1 & (\alpha,\beta) = (-1/2, -1/2) \\ \operatorname{sign}(\cos(\theta/2)), & (\alpha,\beta) = (-1/2, 0) \\ \operatorname{sign}(\sin(\theta/2)), & (\alpha,\beta) = (0, -1/2) \end{cases}$$

and

$$\varphi(x,t) := \arccos x + \arccos t, \qquad \psi(x,t) := \arccos x - \arccos t.$$

Lemma 5. *Let* $(\alpha, \beta) \in \{(0, 0), (-1/2, -1/2), (-1/2, 0), (0, -1/2)\}$, *and* $Y_n := \operatorname{span}\{P_{k_j}^{2\alpha+1/2,\,2\beta+1/2}(x);\ k_j \in I_n\}$. *For an arbitrary projection operator* $P_{I_n}: L^1_{w(\alpha,\beta)} \to Y_n$ *there hold the Berman–Marcinkiewicz type identity*

$$\frac{2}{\pi}\int_{-1}^{1} (\tau_t^{\alpha,\beta} P_{I_n} \tau_t^{\alpha,\beta} f)(x) \frac{dt}{\sqrt{1-t^2}}$$

$$= \begin{cases} (S_{I_n}^{-1/2,\,-1/2} f)(x) + (S_0^{-1/2,\,-1/2} f)(x), & (\alpha,\beta) = (-1/2,-1/2) \text{ and } 0 \in I_n \\ (S_{I_n}^{2\alpha+1/2,\,2\beta+1/2} f)(x), & \text{otherwise} \end{cases} \tag{9}$$

for $f \in L^1_{w(\alpha,\beta)}$ *and the inequality*

$$\begin{aligned} \| S_{I_n}^{2\alpha+1/2,\, 2\beta+1/2} \|_{[L^1_{w(\alpha,\beta)}]} &\geqslant \lambda(Y_n, L^1_{w(\alpha,\beta)}) \\ &\geqslant \begin{cases} \frac{1}{2} \| S_{I_n}^{-1/2,\, -1/2} \|_{[L^1_{w(-1/2,\, -1/2)}]} - \frac{1}{2}, & (\alpha,\beta) = (-1/2, -1/2) \text{ and } 0 \in I_n \\ \frac{1}{2} \| S_{I_n}^{2\alpha+1/2,\, 2\beta+1/2} \|_{[L^1_{w(\alpha,\beta)}]}, & \text{otherwise.} \end{cases} \end{aligned} \tag{10}$$

If I is a thin index set, it follows that

$$\lambda(Y_n, L^1_{w(\alpha,\beta)}) \geqslant C\sqrt{n}. \tag{11}$$

By the Theorem of Kadec and Snobar [8, (5)], the inverse inequality is valid for arbitrary index sets with constant $C = 1$.

Outline of Proof. The proof is similar to [5, 6], except for the more general index sets admitted here. To obtain (9) one proceeds like in the trigonometric case [2, p. 282], replacing the product formula $T_t(e^{ik\cdot})(x) = e^{ik(x+t)} = e^{ikx}e^{ikt}$ for the usual translation operator T_t by

$$(\tau_t^{0,0} \tilde{P}_j^{1/2,\, 1/2}(\cdot))(x) = \sqrt{\frac{\pi}{2}}\, \tilde{P}_{j+1}^{-1/2,\, -1/2}(t)\, \tilde{P}_j^{1/2,\, 1/2}(x) \qquad (j \in \mathbb{N}_0),$$

in case $(\alpha, \beta) = (0, 0)$, where $\tau_t^{0,0}$ is given by Definition 4, and observing that it is not necessary that the two factors on the right belong to the same set of orthogonal polynomials. In fact, by means of two isometric isomorphisms the computations are finally done in a trigonometric setting, i.e., in terms of projections from the space $\tilde{L}^1_{2\pi}$ of even integrable and 2π-periodic functions onto $\operatorname{span}\{\sin((k+1)\arccos x);\ 0 \leqslant k \leqslant n\}$. Similar product formulas exist for $(\alpha, \beta) \in \{(-1/2, -1/2), (-1/2, 0), (0, -1/2)\}$.

As to (10), the upper estimate is trivial. The lower estimate follows by (9), the relation

$$\| \tau_t^{\alpha,\beta} \|_{[L^1_{w(\alpha;\beta)}]} \leqslant 1, \tag{12}$$

and elementary estimations of the operator norms. This approach is based on the fact that just in the four cases treated the Jacobi polynomials admit trigonometric representations like (7) and that in space $L^1_{w(\alpha,\beta)}$ the norms of the partial sums $S_n^{2\alpha+1/2,\, 2\beta+1/2}$ increase more slowly than those of the Fourier–Jacobi partial sums $S_n^{\alpha,\beta}$ [3, 4]. ∎

Concerning dense index sets one has the following

LEMMA 6. *Let* $(\alpha, \beta) \in \{(0,0), (-1/2, -1/2), (-1/2, 0), (0, -1/2)\}$ *and let* $I_n^{(m)} = \{m+1, m+2, ..., m+n\}$ *for some* $n \in \mathbb{N}_0$. *There exists a constant* $C > 0$, *independent of m and n, such that*

$$\|S^{2\alpha+1/2,\, 2\beta+1/2}_{I_n^{(m)}}\|_{[L^1_{w(\alpha,\beta)}]} \leqslant C \log n \qquad (n \in \mathbb{N}, n \geqslant 2). \tag{13}$$

Proof. In case $(\alpha, \beta) = (0, 0)$, one has, denoting by D_n the Dirichlet kernel,

$$\begin{aligned}
\|S^{1/2,\,1/2}_{I_n^{(m)}}\|_{[L^1(-1,1)]} &= \frac{1}{\pi} \operatorname*{ess\,sup}_{\psi \in (0,\pi)} \left\| \sum_{j=1}^{n} \sin((m+j+1)\,\psi)\sin((m+j+1)\cdot) \right\|_{L^1_{2\pi}} \\
&\leqslant \frac{1}{2\pi} \|D_{m+n+1} - D_{m+1}\|_{L^1_{2\pi}} \\
&\leqslant \frac{1}{2\pi} \left\| \frac{1}{\sin(\cdot/2)} \sin\left(\frac{2m+3}{2}\cdot\right)[1-\cos(n\cdot)] \right\|_{L^1_{2\pi}} \\
&\quad + \frac{1}{2\pi} \left\| \frac{1}{\sin(\cdot/2)} \cos\left(\frac{2m+3}{2}\cdot\right)\sin(n\cdot) \right\|_{L^1_{2\pi}} \\
&\leqslant \frac{1}{2\pi} \left\| \frac{1-\cos(n\cdot)}{\sin(\cdot/2)} \right\|_{L^1_{2\pi}} + \frac{1}{2\pi} \left\| \frac{\sin(n\cdot)}{\sin(\cdot/2)} \right\|_{L^1_{2\pi}} \\
&= \frac{1}{\pi} \left\| \frac{\sin^2((n/2)\cdot)}{\sin(\cdot/2)} \right\|_{L^1_{2\pi}} + \frac{1}{\pi} \left\| \frac{\sin((n/2)\cdot)\cos((n/2)\cdot)}{\sin(\cdot/2)} \right\|_{L^1_{2\pi}} \\
&\leqslant \frac{2}{\pi} \left\| \frac{\sin((n/2)\cdot)}{\sin(\cdot/2)} \right\|_{L^1_{2\pi}},
\end{aligned}$$

and the proof is completed as in [1, Proposition 1.2.3]. The other three cases can be treated similarly. ▮

Remark 7. By [7, Theorem 1], the inverse of (13) is valid for arbitrary index sets. These inequalities can be transferred to relative projection constants as in Lemma 5.

Proof of the Theorem. In case $X(-1,1) = L^1_{w(\alpha,\beta)}$, the construction of [10] remains applicable by using Lemmas 3, 5, and 6. We outline the main steps. By Lemma 5 it suffices to show that the $S^{2\alpha+1/2,\,2\beta+1/2}_{I_n}$ satisfy the same estimates. We build an index set $I = \{k_j \in \mathbb{N}_0;\ k_1 < k_2 < \cdots\}$ as follows. Given $\gamma \in (0, 1/2]$, set $a := 3/(2\gamma) - 1$, thus $a > 2$. Choose a lacunary index set $L = \{\lambda_j \in \mathbb{N};\ \lambda_1 < \lambda_2 < \cdots\}$ satisfying $\lambda_{m+1} - \lambda_m > [n^a]$ $(m \geqslant n)$. As a first step, take $k_1 := \lambda_1$ from L and $k_2 := k_1 + 1$.

In step 2, let the following 2^2 indices come from L, thus

$$k_3 := \lambda_{1+1} \equiv \lambda_2,\ k_4 := \lambda_{1+2} \equiv \lambda_3,\ ...,\ k_6 := \lambda_{1+4} \equiv \lambda_5,$$

and continue with $[2^a]$ "arithmetic indices":

$$k_7 := \lambda_5 + 1, k_8 := \lambda_5 + 2, ..., k_{6+[2^a]} := \lambda_5 + [2^a].$$

Step j consists in choosing the next j^2 entries from L, thus

$$\lambda_{\nu+1}, \lambda_{\nu+2}, ..., \lambda_{\nu+j^2} \qquad (\nu = \nu(j)),$$

and continuing with $[j^a]$ "arithmetic entries":

$$\lambda_{\nu+j^2} + 1, \lambda_{\nu+j^2} + 2, ..., \lambda_{\nu+j^2} + [j^a].$$

Let

$$I = \bigcup_{j \in \mathbb{N}} (L_j \cup A_j)$$

denote the index set obtained, where $L_j := \{\lambda_{\nu(j)+1}, \lambda_{\nu(j)+2}, ..., \lambda_{\nu(j)+j^2}\}$ are the lacunary parts and $A_j := \{\lambda_{\nu(j)+j^2} + 1, \lambda_{\nu(j)+j^2} + 2, ..., \lambda_{\nu(j)+j^2} + [j^a]\}$ the arithmetic ones. By construction we have

$$\# L_j = j^2 \qquad \text{and} \qquad \# A_j = [j^a] \qquad (j \in \mathbb{N}), \tag{14}$$

and it can be shown, given $n \in \mathbb{N}$ and supposing that $k_n \in I$ was obtained during the $(M+1)$st step, that there exist constants C_1, C_2 such that $C_1 M \leqslant n^{1/(a+1)} \leqslant C_2 M$. Define $L_{M+1}^{(n)} \subset L_{M+1}$ and $A_{M+1}^{(n)} \subset A_{M+1}$ by

$$I_n = \{k_1, k_2, ..., k_n\} = \bigcup_{j=1}^{M} (L_j \cup A_j) \cup L_{M+1}^{(n)} \cup A_{M+1}^{(n)}.$$

Then (14) implies

$$\# L_{M+1}^{(n)} \leqslant (M+1)^2 \qquad \text{and} \qquad \# A_{M+1}^{(n)} \leqslant [(M+1)^a]. \tag{15}$$

The proof proceeds by observing that the Jacobi partial sums $S^{2\alpha+1/2, 2\beta+1/2}_{\bigcup_{j=1}^{M} L_j \cup L_{M+1}^{(n)}}$, $S^{2\alpha+1/2, 2\beta+1/2}_{A_j}$, $1 \leqslant j \leqslant M$, and $S^{2\alpha+1/2, 2\beta+1/2}_{A_{M+1}^{(n)}}$ are pairwise orthogonal, so that the following decomposition holds

$$\begin{aligned} S^{2\alpha+1/2, 2\beta+1/2}_{I_n} &= S^{2\alpha+1/2, 2\beta+1/2}_{\bigcup_{j=1}^{M} L_j \cup L_{M+1}^{(n)}} + \sum_{j=1}^{M} S^{2\alpha+1/2, 2\beta+1/2}_{A_j} + S^{2\alpha+1/2, 2\beta+1/2}_{A_{M+1}^{(n)}} \\ &:= \sigma_1 + \sigma_2 + \sigma_3, \end{aligned}$$

say, and carefully estimating the norm of the latter sum from above and below. An upper bound is obtained by using Lemma 6 for Fourier–Jacobi partial sums on arithmetic sections, so that

$$\| S^{2\alpha+1/2, 2\beta+1/2}_{A_j} \|_{[L^1_{w(\alpha,\beta)}]} \leqslant C \log(\# A_j),$$

and hence

$$\|\sigma_2\| + \|\sigma_3\| \leqslant C \sum_{j=1}^{M} \log[j^\alpha] \leqslant C(M+1)\log(M+1),$$

and on the other hand, by (15), (14), and the Kadec–Snobar theorem,

$$\|\sigma_1\| \leqslant C \sqrt{\#\left(\bigcup_{j=1}^{M} L_j \cup L_{M+1}^{(n)}\right)} \leqslant C \sqrt{\sum_{j=1}^{M+1} j^2} \leqslant CM^{3/2}.$$

Hence the term σ_1, which represents the sparse sections of the index set, supersedes σ_2 and σ_3. By the properties of a, $M^{3/2}$ behaves like n^γ. The lower bound is obtained similarly, using Lemma 3.

In case $X(-1,1) = C[-1,1]$, the assertion is an immediate consequence of the case $(\alpha, \beta) = (-1/2, -1/2)$ and part (c) of the following.

LEMMA 8. *Let* $Y_n := \operatorname{span}\{P_{k_j}^{-1/2, -1/2}(x); k_j \in I_n\}$, $n \in \mathbb{N}$.

(a) $$\|S_{I_n}^{-1/2, -1/2}\|_{[C[-1,1]]} \geqslant \lambda(Y_n, C[-1,1])$$
$$\geqslant \begin{cases} \frac{1}{2}\|S^{-1/2,-1/2}\|_{[C[-1,1]]} - \frac{1}{2}, & 0 \in I_n \\ \frac{1}{2}\|S_{I_n}^{-1/2,-1/2}\|_{[C[-1,1]]}, & 0 \notin I_n, \end{cases}$$

(b) $\|S_{I_n}^{-1/2,-1/2}\|_{[C[-1,1]]} = \|S_{I_n}^{-1/2,-1/2}\|_{[L^1_{w(-1/2,-1/2)}]}$,

(c) *There exist constants* $C_1, C_2 > 0$ *such that for each* $n \in \mathbb{N}$

$$C_1 \lambda(Y_n, C[-1,1]) \leqslant \lambda(Y_n, L^1_{w(-1/2,-1/2)}) \leqslant C_2 \lambda(Y_n, C[-1,1]).$$

Proof. If $L^1_{w(-1/2,-1/2)}$ is replaced by $C[-1,1]$ in Lemma 5, the Berman–Marcinkiewicz type identity of (9) holds for arbitrary bounded linear projections $P_{I_n}: C[-1,1] \to Y_n$, and the bound $\|\tau_t^{-1/2,-1/2}\|_{[C[-1,1]]} \leqslant 1$ for the Chebyshev translation operator $\tau_t^{-1/2,-1/2}$ yields (a) (cf. (10)). Parts (b) and (c) are obvious. ∎

Remark 9. The construction in [10]—and thus the theorem—is also valid for $\gamma = 1/2$ and furnishes an example of a non-thin index set with property (11).

Remark 10. We conclude with an example showing that increasing the growth rate of the k_j in (3) does not necessarily result in an acceleration of the growth of $\lambda(Y_n, X(-1,1))$ in (2).

By Proposition 2 and Lemma 5, a lacunary index set of the form $\{k_j := 2^j; j \in \mathbb{N}\}$ leads to a growth rate $\sqrt{n}$. On the other hand, as in [10], choosing $a = 5$ there, an index set $I' = \{k'_j \in \mathbb{N};\ k'_1 < k'_2 < \cdots\}$ can be constructed which leads to a growth rate $n^{1/4}$. Proceeding as in [10] and choosing $\lambda_j := j^{j^2}$ which satisfies the hypothesis [10, (5)], one obtains an index set $M \equiv I'$ with

$$\liminf_{j \to \infty} \frac{k'_j}{k_j} = +\infty.$$

REFERENCES

1. P. L. Butzer and R. J. Nessel, "Fourier-Analysis and Approximation," Vol. 1, Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Mathematische Reihe, Vol. 40, Birkhäuser, Basel, 1971.
2. R. A. DeVore and G. G. Lorentz, "Constructive Approximation," Grundlehren Math. Wiss., Vol. 303, Springer-Verlag, Berlin, 1993.
3. E. Görlich and C. Markett, Projections with norms smaller than those of the ultraspherical and Laguerre partial sums, *in* "Functional Analysis and Approximation" (P. L. Butzer, B. Sz. Nagy, and E. Görlich, Eds.), Internat. Ser. Numer. Math., Vol. 60, pp. 189–202, Birkhäuser, Basel, 1981.
4. E. Görlich and C. Markett, A lower bound for projection operators on $L^1(-1, 1)$, *Ark. Mat.* **24** (1986), 81–92.
5. E. Görlich and A. P. Rohs, Asymptotic minimal projection constants for Lebesgue spaces, *C. R. Math. Rep. Acad. Sci. Canada* **16** (1994), 15–16.
6. E. Görlich and A. P. Rohs, Asymptotic bounds for projection operators on Lebesgue spaces, *Integral Transforms Special Functions* **3** (1995), 85–98.
7. E. Görlich and A. P. Rohs, Bounds for relative projection constants in $L^1(-1, 1)$, *Constr. Approx.* **14** (1998), 589–597.
8. M. I. Kadec and M. G. Snobar, Some functionals over a compact Minkovskii space, *Mat. Zametki* **10** (1971), 453–457; English translation, *Math. Notes*, 694–696.
9. D. J. Newman and B. Shekhtman, A Losynski–Kharshiladze theorem for Müntz polynomials, *Acta Math. Hungar.* **45** (1985), 301–303.
10. B. Shekhtman, On one problem of G. G. Lorentz regarding the norms of Fourier projections, *Proc. Amer. Math. Soc.* **108** (1990), 187–190.